

# The Local Cut Lemma

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## Abstract

The Lovász Local Lemma is a very powerful tool in probabilistic combinatorics, that is often used to prove existence of combinatorial objects satisfying certain constraints. Moser and Tardos [25] have shown that the LLL gives more than just pure existence results: there is an effective randomized algorithm that can be used to find a desired object. In order to analyze this algorithm, Moser and Tardos developed the so-called *entropy compression method*. It turned out that one could obtain better combinatorial results by a direct application of the entropy compression method rather than simply appealing to the LLL. The aim of this paper is to provide a generalization of the LLL which implies these new combinatorial results. This generalization, which we call the Local Cut Lemma, concerns a random cut in a directed graph with certain properties. Note that our result has a short probabilistic proof that does not use entropy compression. As a consequence, it not only shows that a certain probability is positive, but also gives an explicit lower bound for this probability. As an illustration, we present a new application (an improved lower bound on the number of edges in color-critical hypergraphs) as well as explain how to use the Local Cut Lemma to derive some of the results obtained previously using the entropy compression method.

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## 1 Introduction

One of the most useful tools in probabilistic combinatorics is the so-called *Lovász Local Lemma* (the LLL for short), which was proved by Erdős and Lovász in their seminal paper [11]. Roughly speaking, the LLL asserts that, given a family  $\mathcal{B}$  of random events whose individual probabilities are small and whose dependency is somehow limited, there is a positive probability that none of the events in  $\mathcal{B}$  happen. More precisely:

**Theorem 1** (Lovász Local Lemma, [6]). *Let  $\mathcal{B}$  be a finite family of random events in a probability space  $\Omega$ . For each  $B \in \mathcal{B}$ , let  $\Gamma(B)$  be a subset of  $\mathcal{B} \setminus \{B\}$  such that  $B$  is independent from the algebra generated by  $\mathcal{B} \setminus (\Gamma(B) \cup \{B\})$ . Suppose that there exists an assignment of reals  $\mu : \mathcal{B} \rightarrow [0; 1]$  such that for every  $B \in \mathcal{B}$  we have*

$$\Pr(B) \leq \mu(B) \prod_{B' \in \Gamma(B)} (1 - \mu(B')).$$

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Then

$$\Pr \left( \bigcap_{B \in \mathcal{B}} \overline{B} \right) \geq \prod_{B \in \mathcal{B}} (1 - \mu(B)) > 0.$$

Note that the probability  $\Pr \left( \bigcap_{B \in \mathcal{B}} \overline{B} \right)$ , which the LLL bounds from below, is usually exponentially small (in the number of events in  $\mathcal{B}$ ). This is in contrast to the more common situation in the probabilistic method when the probability of interest is not only positive, but separated from zero. Although this property of the LLL makes it an indispensable tool in proving combinatorial existence results, it also makes these results seemingly nonconstructive, since sampling the probability space to find an object with the desired properties would usually take an exponentially long expected time. A major breakthrough was made by Moser and Tardos [25], who showed that, in a special framework for the LLL called the *variable version* (the name is due to Kolipaka and Szegedy [21]), there exists a simple Las Vegas algorithm with expected polynomial runtime that searches the probability space for a point which avoids all the events in  $\mathcal{B}$ . Their algorithm was subsequently refined and extended to other situations by several authors; see e.g. [27], [21], [3], [9].

The key ingredient of Moser and Tardos's proof is the so-called *entropy compression method* (the name is due to Tao [29]). The idea of this method is to encode the execution process of the algorithm in such a way that the original sequence of random inputs can be uniquely recovered from the resulting encoding. One then proceeds to show that if the algorithm runs for too long, the space of possible codes becomes smaller than the space of inputs, which leads to a contradiction.

It was discovered lately (and somewhat unexpectedly) that applying the entropy compression method directly can often produce better combinatorial results than simply using the LLL. The idea, first introduced by Grytczuk, Kozik, and Micek in their study of nonrepetitive sequences [18], is to construct a randomized procedure that solves a given combinatorial problem and then apply the entropy compression argument to show that it runs in expected finite time. A wealth of new results have been obtained using this paradigm; see e.g. [10], [13], [15]. Some of these examples are discussed in more detail in Section 4.

Note that the entropy compression method is indeed a “method” that one can use to attack a problem rather than a general theorem that contains various combinatorial results as its special cases. It is natural to ask if such a theorem exists, i.e., if there is a generalization of the LLL that implies the new combinatorial results obtained using the entropy compression method. The goal of this paper is to provide such a generalization, which we call the *Local Cut Lemma* (the LCL for short). It is important to note that this result is purely probabilistic and similar to the LLL in flavor. In particular, its short and simple probabilistic proof does not use the entropy compression method. Instead, it estimates certain probabilities explicitly, in much the same way as the original (nonconstructive) proof of the LLL does.

To state our main result, we need to fix some notation and terminology. In what follows, a *digraph* always means a finite<sup>1</sup> directed multigraph. Let  $D$  be a digraph with vertex set  $V$  and edge set  $E$ . For  $x, y \in V$ , let  $E(x, y) \subseteq E$  denote the set of all edges with tail  $x$  and head  $y$ .

A digraph  $D$  is *simple* if for all  $x, y \in V$ ,  $|E(x, y)| \leq 1$ . If  $D$  is simple and  $|E(x, y)| = 1$ , then the unique edge with tail  $x$  and head  $y$  is denoted by  $xy$  (or sometimes  $(x, y)$ ). For an arbitrary digraph  $D$ , let  $D^s$  denote its *underlying simple digraph*, i.e., the simple digraph with vertex set  $V$  in which  $xy$  is an edge if and only if  $E(x, y) \neq \emptyset$ . Denote the edge set of  $D^s$  by  $E^s$ . For a set  $F \subseteq E$ , let  $F^s \subseteq E^s$  be the set of all edges  $xy \in E^s$  such that  $F \cap E(x, y) \neq \emptyset$ .

A set  $A \subseteq V$  is *out-closed* (resp. *in-closed*) if for all  $xy \in E^s$ ,  $x \in A$  implies  $y \in A$  (resp.  $y \in A$  implies  $x \in A$ ).

**Definition 2.** Let  $D$  be a digraph with vertex set  $V$  and edge set  $E$  and let  $A \subseteq V$  be an out-closed set of vertices. A set  $F \subseteq E$  of edges is an *A-cut* if  $A$  is in-closed in  $D^s - F^s$ .

In other words, an  $A$ -cut  $F$  has to contain at least one edge  $e \in E(x, y)$  for each pair  $x, y$  such that  $x \in V \setminus A$ ,  $y \in A$ , and  $xy \in E^s$ .

We say that a vertex  $z \in V$  is *reachable* from  $x \in V$  if  $D$  (or, equivalently,  $D^s$ ) contains a directed  $xz$ -path. Finally, we introduce two more pieces of notation.

**Definition 3.** Let  $D$  be a digraph with vertex set  $V$  and edge set  $E$ . Suppose that  $\omega : E^s \rightarrow \mathbb{R}_{\geq 1}$  is an assignment of nonnegative real numbers to the edges of  $D^s$ . For  $x, z \in V$  such that  $z$  is reachable from  $x$ , define

$$\underline{\omega}(x, z) := \min \left\{ \prod_{i=1}^k \omega(z_{i-1} z_i) : x = z_0 \longrightarrow z_1 \longrightarrow \dots \longrightarrow z_k = z \text{ is a directed } xz\text{-path in } D^s \right\}.$$

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<sup>1</sup>In fact, the LCL applies, with essentially the same proof, to countably infinite digraphs as well.

**Definition 4.** Let  $D$  be a digraph with vertex set  $V$  and edge set  $E$ . Suppose that  $A \subseteq V$  is a random out-closed set of vertices and let  $F \subseteq E$  be a random  $A$ -cut. Fix a function  $\omega : E^s \rightarrow \mathbb{R}_{\geq 1}$ . For  $xy \in E^s$ ,  $e \in E(x, y)$ , and a vertex  $z \in V$  reachable from  $y$ , let

$$\rho_\omega(e, z) := \Pr(e \in F | z \in A) \cdot \underline{\omega}(x, z). \quad (1)$$

For  $e \in E(x, y)$ , define the *risk* to  $e$  as

$$\rho_\omega(e) := \min_z \rho_\omega(e, z),$$

where  $z$  ranges over all vertices reachable from  $y$ .

For random events  $B, B'$ , the conditional probability  $\Pr(B'|B)$  is only defined if  $\Pr(B) > 0$ . For convenience, we adopt the following notational convention in Definition 4: If  $B$  is a random event and  $\Pr(B) = 0$ , then  $\Pr(B'|B) = 0$  for all events  $B'$ . Note that this way the crucial equation  $\Pr(B'|B) \cdot \Pr(B) = \Pr(B' \cap B)$  is satisfied even when  $\Pr(B) = 0$ , and this is the only property of conditional probability we will use.

We are now ready to state the main result of this paper.

**Theorem 5** (Local Cut Lemma). *Let  $D$  be a digraph with vertex set  $V$  and edge set  $E$ . Suppose that  $A \subseteq V$  is a random out-closed set of vertices and let  $F \subseteq E$  be a random  $A$ -cut. If a function  $\omega : E^s \rightarrow \mathbb{R}_{\geq 1}$  satisfies the following inequality for all  $xy \in E^s$ :*

$$\omega(xy) \geq 1 + \sum_{e \in E(x, y)} \rho_\omega(e), \quad (2)$$

then for all  $xy \in E^s$ ,

$$\Pr(y \in A) \leq \Pr(x \in A) \cdot \omega(xy).$$

The following immediate corollary is the main tool used in combinatorial applications of Theorem 5.

**Corollary 6.** *Let  $D, A, F, \omega$  be as in Theorem 5. Let  $x, z \in V$  be such that  $z$  is reachable from  $x$  and suppose that  $\Pr(z \in A) > 0$ . Then*

$$\Pr(x \in A) \geq \frac{\Pr(z \in A)}{\underline{\omega}(x, z)} > 0.$$

One particular class of digraphs often appearing in applications of the LCL is the class of the *hypercube digraphs*. For a finite set  $I$ , the vertices of the hypercube digraph  $Q(I)$  are the subsets of  $I$  and its edges are of the form  $(S \cup \{i\}, S)$  for each  $S \subset I$  and  $i \in I \setminus S$  (in particular,  $Q(I)$  is simple). Note that if  $S, T \subseteq I$ , then  $T$  is reachable from  $S$  in  $Q(I)$  if and only if  $T \subseteq S$ . Moreover, if  $T \subseteq S$ , then all directed  $(S, T)$ -paths have length exactly  $|S \setminus T|$ . Therefore, if  $\omega(S \cup \{i\}, S) = \omega(i)$  only depends on  $i$ , then for all  $T \subseteq S \subseteq I$ ,

$$\underline{\omega}(S, T) = \prod_{i \in S \setminus T} \omega(i).$$

If we further assume that  $\omega(S \cup \{i\}, S) = \omega \in \mathbb{R}_{\geq 1}$  is a fixed constant, then

$$\underline{\omega}(S, T) = \omega^{|S \setminus T|}.$$

Although the frameworks of the LCL and the LLL seem somewhat dissimilar, the LLL can be viewed as a special case of the LCL, as we explain in Subsection 2.1. In Subsection 2.2 we describe in detail one simple example (namely hypergraph coloring), which provides the intuition behind the LCL and serves as a model for the more substantial applications we discuss later. Section 3 contains the proof of the LCL. Section 4 is dedicated to combinatorial applications of the lemma. There we explain how to use the LCL to prove several results obtained previously using the entropy compression method. We also present a new application (an improved lower bound on the number of edges in color-critical hypergraphs) in Subsection 4.3. The last application, discussed in Subsection 4.4, is a curious probabilistic corollary of the LCL.

## 2 First examples

### 2.1 The LCL implies the Lopsided LLL

In this subsection we use the LCL to prove the Lopsided LLL, which is a strengthening of the standard LLL.

**Theorem 7** (Lopsided Lovász Local Lemma, [12]). *Let  $\mathcal{B}$  be a finite family of random events in a probability space  $\Omega$ . For each  $B \in \mathcal{B}$ , let  $\Gamma(B)$  be a subset of  $\mathcal{B} \setminus \{B\}$  such that*

$$\Pr(B) \geq \Pr\left(B \mid \bigcap_{B' \in \mathcal{S}} \overline{B'}\right) \quad (3)$$

*for every  $\mathcal{S} \subseteq \mathcal{B} \setminus (\Gamma(B) \cup \{B\})$ . Suppose that there exists an assignment of reals  $\mu : \mathcal{B} \rightarrow [0; 1)$  such that for every  $B \in \mathcal{B}$  we have*

$$\Pr(B) \leq \mu(B) \prod_{B' \in \Gamma(B)} (1 - \mu(B')). \quad (4)$$

*Then*

$$\Pr\left(\bigcap_{B \in \mathcal{B}} \overline{B}\right) \geq \prod_{B \in \mathcal{B}} (1 - \mu(B)) > 0.$$

*Proof.* We will apply the LCL to the hypercube digraph  $Q := Q(\mathcal{B})$ . Let  $A \subseteq V(Q)$  be the set of vertices of  $Q$  such that

$$\mathcal{S} \in A : \iff \bigcap_{B \in \mathcal{S}} \overline{B} \text{ happens.}$$

Note that  $A$  is out-closed. Moreover,  $\emptyset \in A$  with probability 1. Since  $\emptyset$  is reachable from  $\mathcal{B}$  in  $Q$ , if we can successfully apply the LCL here, Corollary 6 will imply that  $\Pr(\mathcal{B} \in A) > 0$ . But  $\mathcal{B} \in A$  if and only if none of the events in  $\mathcal{B}$  happen, and our goal is exactly to show that the probability of this event is positive.

Let  $F$  be the set of edges of  $Q$  such that

$$(\mathcal{S} \cup \{B\}, \mathcal{S}) \in F : \iff B \text{ happens.}$$

Note that if  $\mathcal{S} \in A$ , but  $\mathcal{S} \cup \{B\} \notin A$ , then  $B$  happens, and so  $(\mathcal{S} \cup \{B\}, \mathcal{S}) \in F$ . In other words,  $F$  is an  $A$ -cut.

Assume that  $\omega(\mathcal{S} \cup \{B\}, \mathcal{S}) = \omega(B)$  only depends on  $B$ . We want to get an upper bound on the risk to an edge  $(\mathcal{S} \cup \{B\}, \mathcal{S})$  of  $Q$ . For this, we need to find a vertex  $\mathcal{Z}$  of  $Q$  reachable from  $\mathcal{S}$  (i.e., a subset  $\mathcal{Z} \subseteq \mathcal{S}$ ) such that the following probability is easy to estimate:

$$\Pr((\mathcal{S} \cup \{B\}, \mathcal{S}) \in F \mid \mathcal{Z} \in A), \quad (5)$$

since this is the probability appearing in (1). Using the definitions of  $A$  and  $F$ , we can rewrite (5) as

$$\Pr\left(B \mid \bigcap_{B' \in \mathcal{Z}} \overline{B'}\right).$$

This expression looks similar to the right-hand side of (3). Keeping that in mind, consider  $\mathcal{Z} := \mathcal{S} \setminus \Gamma(B)$ . Then

$$\Pr\left(B \mid \bigcap_{B' \in \mathcal{Z}} \overline{B'}\right) = \Pr\left(B \mid \bigcap_{B' \in \mathcal{S} \setminus \Gamma(B)} \overline{B'}\right) \leq \Pr(B).$$

Since

$$(\mathcal{S} \cup \{B\}) \setminus \mathcal{Z} = (\mathcal{S} \cup \{B\}) \setminus (\mathcal{S} \setminus \Gamma(B)) \subseteq \Gamma(B) \cup \{B\},$$

we get

$$\underline{\omega}(\mathcal{S} \cup \{B\}, \mathcal{Z}) \leq \prod_{B' \in \Gamma(B) \cup \{B\}} \omega(B').$$

Therefore,

$$\rho_\omega((\mathcal{S} \cup \{B\}, \mathcal{S})) \leq \rho_\omega((\mathcal{S} \cup \{B\}, \mathcal{S}), \mathcal{Z}) \leq \Pr(B) \prod_{B' \in \Gamma(B) \cup \{B\}} \omega(B').$$

Thus, to apply the LCL, it is enough to find a function  $\omega : \mathcal{B} \rightarrow \mathbb{R}_{\geq 1}$  such that for all  $B \in \mathcal{B}$ ,

$$\omega(B) \geq 1 + \Pr(B) \prod_{B' \in \Gamma(B) \cup \{B\}} \omega(B').$$

We claim that  $\omega(B) := 1/(1 - \mu(B))$  is such a function. Indeed, for this  $\omega$ ,

$$\begin{aligned} 1 + \Pr(B) \prod_{B' \in \Gamma(B) \cup \{B\}} \omega(B') &= 1 + \frac{\Pr(B)}{\prod_{B' \in \Gamma(B) \cup \{B\}} (1 - \mu(B'))} \\ &\leq 1 + \frac{\mu(B) \prod_{B' \in \Gamma(B)} (1 - \mu(B'))}{\prod_{B' \in \Gamma(B) \cup \{B\}} (1 - \mu(B'))} \\ &= 1 + \frac{\mu(B)}{1 - \mu(B)} = \frac{1}{1 - \mu(B)} = \omega(B), \end{aligned}$$

where the inequality is due to (4). Corollary 6 now implies that

$$\Pr\left(\bigcap_{B \in \mathcal{B}} \overline{B}\right) = \Pr(\mathcal{B} \in A) \geq \frac{\Pr(\emptyset \in A)}{\underline{\omega}(\mathcal{B}, \emptyset)} = \frac{1}{\prod_{B \in \mathcal{B}} \omega(B)} = \prod_{B \in \mathcal{B}} (1 - \mu(B)),$$

as desired. ■

## 2.2 Hypergraph coloring

Let  $\mathcal{H}$  be a  $d$ -regular  $k$ -uniform hypergraph and suppose we want to establish a relation between  $d$  and  $k$  that guarantees that  $\mathcal{H}$  is 2-colorable. A straightforward application of the LLL gives the bound

$$\frac{e}{2^{k-1}}((d-1)k+1) \leq 1,$$

which is equivalent to

$$d \leq \frac{2^{k-1}}{ek} + 1 - \frac{1}{k}. \quad (6)$$

Let us explain how to apply the LCL in this situation. We start by choosing a coloring  $\varphi : V(\mathcal{H}) \rightarrow \{\text{red}, \text{blue}\}$  uniformly at random. Consider the hypercube digraph  $Q := Q(V(\mathcal{H}))$ . Let  $A \subseteq V(Q)$  be given by

$$S \in A :\iff \text{there is no } \varphi\text{-monochromatic edge } H \subseteq S.$$

Notice that  $A$  is out-closed. Additionally,  $\emptyset \in A$  with probability 1, while  $V(\mathcal{H}) \in A$  if and only if  $\varphi$  is a proper coloring of  $\mathcal{H}$ .

Consider an arbitrary edge  $(S \cup \{v\}, S)$  of  $Q$ . Observe that if  $S \in A$ , but  $S \cup \{v\} \notin A$ , then there is a  $\varphi$ -monochromatic edge  $H \in E(\mathcal{H})$  such that  $v \in H \subseteq S \cup \{v\}$ . This observation motivates the following definition. Let  $\tilde{Q}$  be the digraph such that  $\tilde{Q}^s = Q$  and for each  $(S \cup \{v\}, S) \in E(Q)$  and  $H \in E(\mathcal{H})$  such that  $v \in H \subseteq S \cup \{v\}$ , there is a corresponding edge  $e_H \in E(\tilde{Q})$  going from  $S \cup \{v\}$  to  $S$ . Note that, if  $E := E(\tilde{Q})$ , we have

$$|E(S \cup \{v\}, S)| \leq d, \quad (7)$$

since  $\mathcal{H}$  is  $d$ -regular. Now we can let

$$e_H \in F :\iff H \text{ is } \varphi\text{-monochromatic}.$$

Then  $F$  is an  $A$ -cut.

Assume that  $\omega \in \mathbb{R}_{\geq 1}$  is a constant. Let us estimate  $\rho_\omega(e_H)$  for each edge  $e_H \in E(S \cup \{v\}, S)$ . Since the colors of distinct vertices are independent, we can consider the set  $S \setminus H$ , for which we have

$$\Pr(e_H \in F | S \setminus H \in A) \leq \Pr(e_H \in F) = \frac{1}{2^{k-1}}. \quad (8)$$

(Here we might have a strict inequality because if  $\Pr(S \setminus H \in A) = 0$ , then  $\Pr(e_H \in F | S \setminus H \in A) = 0$  as well, due to our convention regarding conditional probabilities.) As we discussed previously, for all  $T \subseteq S \subseteq V(H)$ ,

$$\underline{\omega}(S, T) = \omega^{|S \setminus T|}.$$

In particular,

$$\underline{\omega}(S \cup \{v\}, S \setminus H) = \omega^{|H|} = \omega^k. \quad (9)$$

Therefore,

$$\rho_\omega(e_H) \leq \rho_\omega(e_H, S \setminus H) \leq \frac{1}{2^{k-1}} \cdot \omega^k.$$

Hence, it is enough to ensure that  $\omega$  satisfies

$$\omega \geq 1 + d \cdot \frac{1}{2^{k-1}} \cdot \omega^k.$$

A straightforward calculation shows that the following condition is sufficient:

$$d \leq \frac{2^{k-1}}{k} \left(1 - \frac{1}{k}\right)^{k-1}, \quad (10)$$

or, a bit more crudely,

$$d \leq \frac{2^{k-1}}{ek}, \quad (11)$$

which is almost identical to (6). Note that the precise bound (10) is, in fact, strictly better than (6) for  $k \geq 10$ .

We can improve (11) slightly by estimating  $\rho_\omega(e_H)$  more carefully. Notice that the probability of a given edge  $H$  being  $\varphi$ -monochromatic remains the same even if the color of one of its vertices is fixed. For  $e_H \in E(S \cup \{v\}, S)$ , choose an arbitrary vertex  $u \in H \setminus \{v\}$  and let  $H' := H \setminus \{u\}$ . Now consider the set  $S \setminus H'$ . We still have

$$\Pr(e_H \in F | S \setminus H' \in A) \leq \frac{1}{2^{k-1}},$$

but now

$$\underline{\omega}(S \cup \{v\}, S \setminus H') = \omega^{|H'|} = \omega^{k-1}.$$

Therefore,

$$\rho_\omega(e_H) \leq \rho_\omega(e_H, S \setminus H') \leq \frac{1}{2^{k-1}} \cdot \omega^{k-1},$$

so it is enough to ensure that

$$\omega \geq 1 + d \cdot \frac{1}{2^{k-1}} \cdot \omega^{k-1},$$

which can be satisfied as long as

$$d \leq \frac{2^{k-1}}{e(k-1)}. \quad (12)$$

The bound (12) is better than (11) by a quantity of order  $\Omega(2^k/k^2)$ . This is, of course, not a significant improvement (and the bound is still considerably weaker than the best known result due to Radhakrishnan and Srinivasan [28], namely  $d \leq \epsilon 2^k / \sqrt{k \log k}$  for some absolute constant  $\epsilon > 0$ ), but it is worth noting that we obtained it virtually “for free”. The trick that we used here will reappear several times in applications discussed in Section 4.

### 3 Proof of the LCL

In this section we prove the LCL. Let  $D, A, F$  be as in Theorem 5 and assume that a function  $\omega : E^s \rightarrow \mathbb{R}_{\geq 1}$  satisfies

$$\omega(xy) \geq 1 + \sum_{e \in E(x,y)} \rho_\omega(e) \quad (2)$$

for all  $xy \in E^s$ . For each  $v : E^s \rightarrow \mathbb{R}_{\geq 1}$ , let  $f(v) : E^s \rightarrow \mathbb{R}_{\geq 1}$  be defined by

$$f(v)(xy) := 1 + \sum_{e \in E(x,y)} \rho_v(e).$$

Also, let  $f(0) := 1$ , where  $0$  and  $1$  denote the constant 0 and 1 functions respectively. Then (2) is equivalent to

$$\omega(xy) \geq f(\omega)(xy).$$

Note that the map  $f$  is monotone increasing, i.e., if  $v(xy) \leq v'(xy)$  for all  $xy \in E^s$ , then  $f(v)(xy) \leq f(v')(xy)$  for all  $xy \in E^s$  as well.

Let  $\omega_0 := 0$  and let  $\omega_{n+1} := f(\omega_n)$  for all  $n \in \mathbb{Z}_{\geq 0}$ . To simplify the notation, let  $\rho_n := \rho_{\omega_n}$ .

**Claim 8.** For all  $n \in \mathbb{Z}_{\geq 0}$  and  $xy \in E^s$ ,

$$\omega_n(xy) \leq \omega_{n+1}(xy). \quad (13)$$

*Proof.* Proof is by induction on  $n$ . If  $n = 0$ , then (13) asserts that  $0 \leq 1$ . Now suppose that (13) holds for some  $n \in \mathbb{Z}_{\geq 0}$ . Then we have

$$\omega_{n+1}(xy) = f(\omega_n)(xy) \leq f(\omega_{n+1})(xy) = \omega_{n+2}(xy),$$

as desired. ■

**Claim 9.** For all  $n \in \mathbb{Z}_{\geq 0}$  and  $xy \in E^s$ ,

$$\omega_n(xy) \leq \omega(xy). \quad (14)$$

*Proof.* Proof is again by induction on  $n$ . If  $n = 0$ , then (14) says that  $0 \leq \omega(xy)$ . Now suppose that (14) holds for some  $n \in \mathbb{Z}_{\geq 0}$ . Then, using (2), we get

$$\omega_{n+1}(xy) = f(\omega_n)(xy) \leq f(\omega)(xy) \leq \omega(xy),$$

as desired. ■

Since the sequence  $\{\omega_n(xy)\}_{n=0}^{\infty}$  is monotone increasing and bounded by  $\omega(xy)$ , it has a limit, so let

$$\omega_{\infty}(xy) := \lim_{n \rightarrow \infty} \omega_n(xy).$$

Note that we still have  $\omega_{\infty}(xy) \leq \omega(xy)$  for all  $xy \in E^s$ . Hence it is enough to prove that for all  $xy \in E^s$ ,

$$\Pr(y \in A) \leq \Pr(x \in A) \cdot \omega_{\infty}(xy). \quad (15)$$

We will derive (15) from the following lemma.

**Lemma 10.** For every  $n \in \mathbb{Z}_{\geq 0}$  and  $xy \in E^s$ ,

$$\Pr(y \in A) \leq \Pr(x \in A) \cdot \omega_n(xy) + \omega_{n+1}(xy) - \omega_n(xy). \quad (16)$$

If Lemma 10 holds, then we are done, since it implies that

$$\Pr(y \in A) \leq \lim_{n \rightarrow \infty} (\Pr(x \in A) \cdot \omega_n(xy) + \omega_{n+1}(xy) - \omega_n(xy)) = \Pr(x \in A) \cdot \omega_{\infty}(xy),$$

as desired.

To establish Lemma 10, we need the following claim.

**Claim 11.** Let  $n \in \mathbb{Z}_{\geq 0}$  and suppose that for all  $xy \in E^s$ , (16) holds. Then for all  $x, z \in V$  such that  $z$  is reachable from  $x$ ,

$$\Pr(z \in A) \leq \Pr(x \in A) \cdot \omega_n(x, z) + \omega_{n+1}(x, z) - \omega_n(x, z). \quad (17)$$

The proof of Claim 11 uses the following simple algebraic inequality.

**Claim 12.** Let  $a_1, \dots, a_k, b_1, \dots, b_k$  be nonnegative real numbers with  $b_i \geq \max\{a_i, 1\}$  for all  $1 \leq i \leq k$ . Then

$$\sum_{i=1}^k \left( \prod_{j=1}^{i-1} a_j \right) (b_i - a_i) \leq \prod_{i=1}^k b_i - \prod_{i=1}^k a_i. \quad (18)$$

*Proof.* Proof is by induction on  $k$ . If  $k = 1$ , then both sides of (18) are equal to  $b_1 - a_1$ . If the claim is established for some  $k$ , then for  $k + 1$  we get

$$\begin{aligned} \sum_{i=1}^{k+1} \left( \prod_{j=1}^{i-1} a_j \right) (b_i - a_i) &= \sum_{i=1}^k \left( \prod_{j=1}^{i-1} a_j \right) (b_i - a_i) + \left( \prod_{i=1}^k a_i \right) b_{k+1} - \prod_{i=1}^{k+1} a_i \\ &\leq \prod_{i=1}^k b_i - \prod_{i=1}^k a_i + \left( \prod_{i=1}^k a_i \right) b_{k+1} - \prod_{i=1}^{k+1} a_i \\ &= \prod_{i=1}^{k+1} b_i - \prod_{i=1}^{k+1} a_i - \left( \prod_{i=1}^k b_i - \prod_{i=1}^k a_i \right) (b_{k+1} - 1) \\ &\leq \prod_{i=1}^{k+1} b_i - \prod_{i=1}^{k+1} a_i, \end{aligned}$$

as desired. ■

*Proof of Claim 11.* Let  $x = z_0 \rightarrow z_1 \rightarrow \dots \rightarrow z_k = z$  be some directed  $xz$ -path in  $D^s$ . For  $1 \leq i \leq k$ , let  $a_i := \omega_n(z_{k-i}z_{k-i+1})$  and  $b_i := \omega_{n+1}(z_{k-i}z_{k-i+1})$ . Note that  $b_i \geq \max\{a_i, 1\}$ .

Due to (16), we have

$$\Pr(z \in A) \leq \Pr(z_{k-1} \in A) \cdot a_1 + b_1 - a_1.$$

Similarly,

$$\Pr(z_{k-1} \in A) \leq \Pr(z_{k-2} \in A) \cdot a_2 + b_2 - a_2,$$

so

$$\Pr(z \in A) \leq \Pr(z_{k-2} \in A) \cdot a_1 a_2 + b_1 - a_1 + a_1(b_2 - a_2).$$

Continuing such substitutions, we finally obtain

$$\Pr(z \in A) \leq \Pr(x \in A) \cdot \prod_{i=1}^k a_i + \sum_{i=1}^k \left( \prod_{j=1}^{i-1} a_j \right) (b_i - a_i).$$

Using Claim 12, we get

$$\Pr(z \in A) \leq \Pr(x \in A) \cdot \prod_{i=1}^k a_i + \prod_{i=1}^k b_i - \prod_{i=1}^k a_i.$$

Note that

$$\prod_{i=1}^k a_i = \prod_{i=1}^k \omega_n(z_{i-1}z_i) \geq \underline{\omega}_n(x, z).$$

Since  $\Pr(x \in A) \leq 1$ , this implies

$$\Pr(z \in A) \leq \Pr(x \in A) \cdot \underline{\omega}_n(x, z) + \prod_{i=1}^k b_i - \underline{\omega}_n(x, z). \quad (19)$$

It remains to observe that inequality (19) holds for all directed  $xz$ -paths, so we can replace  $\prod_{i=1}^k b_i$  by  $\underline{\omega}_{n+1}(x, z)$ , obtaining

$$\Pr(z \in A) \leq \Pr(x \in A) \cdot \underline{\omega}_n(x, z) + \underline{\omega}_{n+1}(x, z) - \underline{\omega}_n(x, z),$$

as desired. ■

*Proof of Lemma 10.* Proof is by induction on  $n$ . For  $n = 0$ , the lemma simply asserts that  $\Pr(y \in A) \leq 1$ . Now assume that (16) holds for some  $n \in \mathbb{Z}_{\geq 0}$  and consider an edge  $xy \in E^s$ . Using the fact that  $F$  is an  $A$ -cut, we get

$$\Pr(y \in A) \leq \Pr(x \in A) + \sum_{e \in E(x, y)} \Pr(e \in F \wedge y \in A).$$

Let us now estimate  $\Pr(e \in F \wedge y \in A)$  for each  $e \in E(x, y)$ . Suppose that  $z \in V$  is a vertex reachable from  $y$ . Since  $A$  is out-closed,  $y \in A$  implies  $z \in A$ , so

$$\Pr(e \in F \wedge y \in A) \leq \Pr(e \in F \wedge z \in A) = \Pr(e \in F | z \in A) \cdot \Pr(z \in A).$$

Due to Lemma 10,

$$\Pr(z \in A) \leq \Pr(x \in A) \cdot \underline{\omega}_n(x, z) + \underline{\omega}_{n+1}(x, z) - \underline{\omega}_n(x, z),$$

so

$$\begin{aligned} \Pr(e \in F \wedge y \in A) &\leq \Pr(e \in F | z \in A) \cdot \left( \Pr(x \in A) \cdot \underline{\omega}_n(x, z) + \underline{\omega}_{n+1}(x, z) - \underline{\omega}_n(x, z) \right) \\ &= \Pr(x \in A) \cdot \rho_n(e, z) + \rho_{n+1}(e, z) - \rho_n(e, z). \end{aligned}$$

Since  $\Pr(x \in A) \leq 1$  and  $\rho_n(e, z) \geq \rho_n(e)$ , we get

$$\Pr(e \in F \wedge y \in A) \leq \Pr(x \in A) \cdot \rho_n(e) + \rho_{n+1}(e, z) - \rho_n(e).$$

The last inequality holds for every  $z$  reachable from  $y$ , so we can replace  $\rho_{n+1}(e, z)$  in it by  $\rho_{n+1}(e)$ , obtaining

$$\Pr(e \in F \wedge y \in A) \leq \Pr(x \in A) \cdot \rho_n(e) + \rho_{n+1}(e) - \rho_n(e).$$



Therefore,

$$\Pr(y \in A) \leq \Pr(x \in A) + \sum_{e \in E(x,y)} (\Pr(x \in A) \cdot \rho_n(e) + \rho_{n+1}(e) - \rho_n(e)).$$

The right hand side of the last inequality can be rewritten as

$$\begin{aligned} & \Pr(x \in A) \cdot \left(1 + \sum_{e \in E(x,y)} \rho_n(e)\right) + \sum_{e \in E(x,y)} \rho_{n+1}(e) - \sum_{e \in E(x,y)} \rho_n(e) \\ &= \Pr(x \in A) \cdot f(\omega_n)(xy) + f(\omega_{n+1})(xy) - f(\omega_n)(xy) \\ &= \Pr(x \in A) \cdot \omega_{n+1}(xy) + \omega_{n+2}(xy) - \omega_{n+1}(xy), \end{aligned}$$

as desired. ■

## 4 Applications

### 4.1 Nonrepetitive sequences and nonrepetitive colorings

A finite sequence  $a_1 a_2 \dots a_n$  is *nonrepetitive* if it does not contain the same nonempty substring twice in a row, i.e., if there are no  $s$ ,  $1 \leq s \leq n-1$ , and  $t$ ,  $1 \leq t \leq \lfloor (n-s+1)/2 \rfloor$ , such that  $a_k = a_{k+t}$  for all  $s \leq k \leq s+t-1$ . A well known result by Thue [30] asserts that there exist arbitrarily long nonrepetitive sequences of elements from  $\{0, 1, 2\}$ . The next theorem is a choosability version of Thue's result. It was the first example of a new combinatorial bound obtained using the entropy compression method that surpasses the analogous bound provided by a direct application of the LLL.

**Theorem 13** (Grytczuk, Przybyło, Zhu [17]; Grytczuk, Kozik, Micek [18]). *Let  $L_1, L_2, \dots, L_n$  be a sequence of sets with  $|L_i| \geq 4$  for all  $1 \leq i \leq n$ . Then there exists a nonrepetitive sequence  $a_1 a_2 \dots a_n$  such that  $a_i \in L_i$  for all  $1 \leq i \leq n$ .*

Note that it is an open problem whether the same result holds for  $|L_i| \geq 3$ .

*Proof.* This is the only example in this paper where the digraph does not come from a hypercube digraph. Let  $P$  be the directed path of length  $n$  with vertex set  $\{1, \dots, n\}$  and with edges of the form  $(i+1, i)$  for all  $1 \leq i \leq n-1$ . Choose a random sequence  $a_1 a_2 \dots a_n$  by selecting each  $a_i \in L_i$  uniformly and independently from each other. Define a set  $A \subseteq V(P)$  as follows:

$$i \in A :\iff a_1 a_2 \dots a_i \text{ is a nonrepetitive sequence.}$$

Note that  $A$  is out-closed,  $\Pr(1 \in A) = 1$ , and  $n \in A$  if and only if  $a_1 a_2 \dots a_n$  is a nonrepetitive sequence.

Consider an edge  $(i+1, i)$  of  $P$ . If  $i \in A$ , but  $i+1 \notin A$ , then there exist  $s$  and  $t$  such that

$$s + 2t - 1 = i + 1$$

and  $a_k = a_{k+t}$  for all  $s \leq k \leq s+t-1$  (i.e.,  $a_s a_{s+1} \dots a_{i+1}$  is a repetition). This observation motivates the following construction. Let  $\tilde{P}$  be the digraph such that  $\tilde{P}^s = P$  and for each  $(i+1, i) \in E(P)$  and  $s, t$  with  $s+2t-1 = i+1$ , there is a corresponding edge  $e_{s,t} \in E(\tilde{P})$  going from  $i+1$  to  $i$ . Let

$$e_{s,t} \in F :\iff a_k = a_{k+t} \text{ for all } s \leq k \leq s+t-1.$$

Then  $F$  is an  $A$ -cut. Note that for each fixed  $t \geq 1$ , there exists at most one  $s$  such that  $s+2t-1 = i+1$ , so there is at most one edge of the form  $e_{s,t} \in E(i+1, i)$ , where  $E$  denotes the edge set of  $\tilde{P}$ .

A vertex  $z \in \{1, \dots, n\}$  is reachable from  $i \in \{1, \dots, n\}$  if and only if  $z \leq i$ . In particular, if  $s+2t-1 = i+1$ , then  $s+t-1$  is reachable from  $i$ . Observe that the probability of  $a_k = a_{k+t}$  is at most  $1/|L_{k+t}|$ , even if the value of  $a_k$  is fixed. Therefore, for  $e_{s,t} \in E(i+1, i)$ , we have

$$\begin{aligned} \Pr(e_{s,t} \in F | s+t-1 \in A) &= \Pr(a_k = a_{k+t} \text{ for all } s \leq k \leq s+t-1 | s+t-1 \in A) \\ &\leq \prod_{k=s}^{s+t-1} \frac{1}{|L_{k+t}|} \leq \frac{1}{4^t}. \end{aligned}$$

If  $\omega(i+1, i) = \omega \in \mathbb{R}_{\geq 1}$  is a fixed constant, then for all  $i \geq j$ ,

$$\underline{\omega}(i, j) = \omega^{i-j}.$$

In particular, if  $s + 2t - 1 = i + 1$ , then

$$\underline{\omega}(i + 1, s + t - 1) = \omega^t.$$

Thus,

$$\rho_\omega(e_{s,t}) \leq \rho_\omega(e_{s,t}, s + t - 1) \leq \frac{1}{4^t} \cdot \omega^t.$$

Hence, it is enough to find a constant  $\omega \in \mathbb{R}_{\geq 1}$  such that

$$\omega \geq 1 + \sum_{t=1}^{\infty} \frac{1}{4^t} \cdot \omega^t = \frac{1}{1 - \omega/4},$$

where the last equality is subject to  $\omega < 4$ . Setting  $\omega = 2$  completes the proof.  $\blacksquare$

A vertex coloring  $\varphi$  of a graph  $G$  is *nonrepetitive* if there is no path  $P$  in  $G$  with an even number of vertices such that the first half of  $P$  receives the same sequence of colors as the second half of  $P$ , i.e., if there is no path  $v_1, v_2, \dots, v_{2t}$  of length  $2t$  such that  $\varphi(v_k) = \varphi(v_{k+t})$  for all  $1 \leq k \leq t$ . The least number of colors that is needed for a nonrepetitive coloring of  $G$  is called the *nonrepetitive chromatic number* of  $G$  and is denoted by  $\pi(G)$ .

The first upper bound on  $\pi(G)$  in terms of the maximum degree  $\Delta(G)$  was given by Alon, Grytczuk, Hałuszczak, and Riordan [4], who proved that there is a constant  $c$  such that  $\pi(G) \leq c\Delta(G)^2$ . Originally this result was obtained with  $c = 2e^{16}$ . The constant was then improved to  $c = 16$  by Grytczuk [19], and then to  $c = 12.92$  by Harant and Jendrol' [20]. All these results were based on the LLL.

Dujmović, Joret, Kozik, and Wood [10] managed to decrease the value of the aforementioned constant  $c$  dramatically using the entropy compression method. Namely, they lowered the constant to 1, or, to be precise, they showed that  $\pi(G) \leq (1 + o(1))\Delta(G)^2$  (assuming  $\Delta(G) \rightarrow \infty$ ).

The currently best known bound is given by the following theorem.

**Theorem 14** (Gonçalves, Montassier, Pinlou [15]). *For every graph  $G$  with maximum degree  $\Delta$ ,*

$$\pi(G) \leq \left\lceil \Delta^2 + \frac{3}{2^{2/3}} \Delta^{5/3} + \frac{2^{2/3} \Delta^{5/3}}{\Delta^{1/3} - 2^{1/3}} \right\rceil.$$

*Proof.* Suppose that

$$k \geq \Delta^2 + \frac{3}{2^{2/3}} \Delta^{5/3} + \frac{2^{2/3} \Delta^{5/3}}{\Delta^{1/3} - 2^{1/3}}. \quad (20)$$

We want to show that  $G$  has a nonrepetitive  $k$ -coloring. Choose a  $k$ -coloring  $\varphi$  of  $G$  uniformly at random. Let  $Q := Q(V(G))$  be the hypercube digraph. Define a set  $A \subseteq V(Q)$  by

$$S \in A :\iff \varphi \text{ is a nonrepetitive coloring of } G[S],$$

where  $G[S]$  denotes the induced subgraph of  $G$  with vertex set  $S$ . Note that  $A$  is out-closed,  $\emptyset \in A$  with probability 1, and  $V(G) \in A$  if and only if  $\varphi$  is a nonrepetitive coloring of  $G$ .

Consider an edge  $(S \cup \{v\}, S)$  of  $Q$ . If  $S \in A$ , but  $S \cup \{v\} \notin A$ , then there exists a path  $P$  of even length that contains  $v$  and is colored repetitively by  $\varphi$ . Let  $\tilde{Q}$  be the digraph such that  $\tilde{Q}^s = Q$  and for each  $(S \cup \{v\}, S) \in E(Q)$  and a path  $P \ni v$  of even length,  $\tilde{Q}$  contains a corresponding edge  $e_P$  going from  $S \cup \{v\}$  to  $S$ . Define

$$e_P \in F :\iff P \text{ is colored repetitively by } \varphi.$$

Then  $F$  is an  $A$ -cut.

Denote the edge set of  $\tilde{Q}$  by  $E$ . Let us estimate the number of edges in  $E(S \cup \{v\}, S)$  corresponding to paths of some fixed length  $2t$ . It is equal to the number of paths  $P$  of length  $2t$  passing through  $v$ . We claim that this number does not exceed  $t\Delta^{2t-1}$ . Indeed, if  $P = v_1, v_2, \dots, v_{2t}$ , then we can assume  $v$  is one of the vertices  $v_1, v_2, \dots, v_t$ , so there are  $t$  ways to choose the position of  $v$  on  $P$ . After the position of  $v$  has been determined, we can select all other vertices one by one so that each time we are choosing only from the neighbors of one of the previous vertices. Since the maximum degree of  $G$  is  $\Delta$ , we get the bound  $t\Delta^{2t-1}$ , as desired.

Now for each  $(S \cup \{v\}, S) \in E^s$  and a path  $P$  of even length passing through  $v$ , we need to upper bound  $\rho_\omega(e_P)$ . Suppose that  $|P| = 2t$ . Let  $P'$  be the half of  $P$  that contains  $v$ . Then

$$\Pr(e_P \in F | S \setminus P' \in A) \leq \frac{1}{k^t},$$

since the coloring of  $P'$  is independent from the coloring of  $S \setminus P'$ . Inasmuch as  $|(S \cup \{v\}) \setminus (S \setminus P')| \leq |P'| = t$ , for a constant  $\omega \in \mathbb{R}_{\geq 1}$ , we have

$$\underline{\omega}(S \cup \{v\}, S \setminus P') \leq \omega^t.$$

Thus,

$$\rho_\omega(e_P) \leq \rho_\omega(e_P, S \setminus P') \leq \frac{\omega^t}{k^t}.$$

Therefore, it is enough to ensure that there exists  $\omega \in \mathbb{R}_{\geq 1}$  such that

$$\omega \geq 1 + \sum_{t=1}^{\infty} t \Delta^{2t-1} \cdot \frac{\omega^t}{k^t} = 1 + \frac{\Delta \omega / k}{(1 - \Delta^2 \omega / k)^2}, \quad (21)$$

where the last equality is subject to  $\Delta^2 \omega / k < 1$ . Setting  $y := \Delta^2 \omega / k$ , we can rewrite (21) as

$$\frac{k}{\Delta^2} \geq \frac{1}{y} + \frac{1}{\Delta(1-y)^2}. \quad (22)$$

Following Gonçalves et al., we take  $y = 1 - (2/\Delta)^{1/3}$ , and (22) becomes

$$\frac{k}{\Delta^2} \geq 1 + \frac{3}{2^{2/3} \Delta^{1/3}} + \frac{2^{2/3}}{\Delta^{2/3} - (2\Delta)^{1/3}},$$

which is true by (20). ■

## 4.2 Acyclic edge colorings

An edge coloring of a graph  $G$  is called an *acyclic edge coloring* if it is proper (i.e. adjacent edges receive different colors) and every cycle in  $G$  contains edges of at least three different colors (there are no *bichromatic cycles* in  $G$ ). The least number of colors needed for an acyclic edge coloring of  $G$  is called the *acyclic chromatic index* of  $G$  and is denoted by  $a'(G)$ . The notion of acyclic (vertex) coloring was first introduced by Grünbaum [16]. The edge version was first considered by Fiamčík [14], and independently by Alon, McDiarmid, and Reed [5].

As in the case of nonrepetitive colorings, it is quite natural to ask for an upper bound on the acyclic chromatic index of a graph  $G$  in terms of its maximum degree  $\Delta(G)$ . Since  $a'(G) \geq \chi'(G) \geq \Delta(G)$ , where  $\chi'(G)$  denotes the ordinary chromatic index of  $G$ , this bound must be at least linear in  $\Delta(G)$ . The first linear bound was given by Alon et al. [5], who showed that  $a'(G) \leq 64\Delta(G)$ . Although it resolved the problem of determining the order of growth of  $a'(G)$  in terms of  $\Delta(G)$ , it was conjectured that the sharp bound should be lower.

**Conjecture 15** (Fiamčík [14]; Alon, Sudakov, Zaks [7]). *For every graph  $G$ ,  $a'(G) \leq \Delta(G) + 2$ .*

Note that the bound in Conjecture 15 is only one more than Vizing's bound on the chromatic index of  $G$ . However, this elegant conjecture is still far from being proven.

The first major improvement to the bound  $a'(G) \leq 64\Delta(G)$  was made by Molloy and Reed [24], who proved that  $a'(G) \leq 16\Delta(G)$ . This bound remained the best for a while, until Ndreca, Procacci, and Scoppola [26] managed to improve it to  $a'(G) \leq \lceil 9.62(\Delta(G) - 1) \rceil$ . Again, first bounds for  $a'(G)$  were obtained using the LLL. The bound  $a'(G) \leq \lceil 9.62(\Delta(G) - 1) \rceil$  by Ndreca et al. used an improved version of the LLL due to Bissacot, Fernández, Procacci, and Scoppola [8].

The best current bound for  $a'(G)$  in terms of  $\Delta(G)$  was obtained by Esperet and Parreau via the entropy compression method.

**Theorem 16** (Esperet, Parreau [13]). *For every graph  $G$  with maximum degree  $\Delta$ ,  $a'(G) \leq 4(\Delta - 1)$ .*

*Proof.* Choose a  $4(\Delta - 1)$ -edge coloring  $\varphi$  of  $G$  uniformly at random. Call a cycle  $C$  of length  $2t$   $\varphi$ -bichromatic if  $C = e_1, e_2, \dots, e_{2t}$  and  $\varphi(e_{2i-1}) = \varphi(e_{2t-1})$ ,  $\varphi(e_{2i}) = \varphi(e_{2t})$  for all  $1 \leq i \leq t-1$ .

Let  $Q := Q(E(G))$  and let

$$S \in A :\iff \varphi \text{ is an acyclic edge coloring of } G[S],$$

where  $G[S]$  is the graph obtained from  $G$  by removing all the edges outside  $S$ . Note that  $A$  is out-closed in  $Q$ ,  $\emptyset \in A$  with probability 1, and  $E(G) \in A$  if and only if  $\varphi$  is an acyclic edge coloring of  $G$ .

Consider an edge  $(S \cup \{e\}, S)$  of  $Q$ . If  $S \in A$ , but  $S \cup \{e\} \notin A$ , then either there exists an edge  $e' \in S$  adjacent to  $e$  such that  $\varphi(e) = \varphi(e')$ , or there exists a  $\varphi$ -bichromatic cycle  $C \subseteq S \cup \{e\}$  of even length such that  $e \in C$ . The crucial idea of [13] (which is credited to Jakub Kozik by the authors) is to handle 4-cycles and cycles of length at least 6 separately. Let  $\tilde{Q}$  be the digraph with  $\tilde{Q}^s = Q$  such that for each  $(S \cup \{e\}, S) \in E(Q)$ , the edges of  $\tilde{Q}$  between  $S \cup \{e\}$  and  $S$  are of the following two kinds:

1.  $f_C$  for each cycle  $C$  of length  $2t \geq 6$  passing through  $e$ ;
2. one additional edge  $f$ .

Let

$$f \in F :\iff \begin{array}{l} \text{either there exists an edge } e' \text{ adjacent to } e \text{ such that } \varphi(e) = \varphi(e'), \\ \text{or there exists a } \varphi\text{-bichromatic 4-cycle } C \ni e, \end{array}$$

and

$$f_C \in F :\iff C \text{ is } \varphi\text{-bichromatic}.$$

Then  $F$  is an  $A$ -cut. Denote the edge set of  $\tilde{Q}$  by  $E$ . Let  $(S \cup \{e\}, S) \in E^s$ , and consider the edge  $f \in E(S \cup \{e\}, S)$  of the second kind. We will estimate the probability

$$\Pr(f \in F | S \in A),$$

using the following claim, which also plays a crucial role in the original proof by Esperet and Parreau.

**Claim 17.** *Suppose that some edges of  $G$  are properly colored. If  $e \in E(G)$  is uncolored, then there exist at most  $2(\Delta - 1)$  ways to color  $e$  so that the resulting coloring either is not proper, or contains a bichromatic 4-cycle going through  $e$ .*

*Proof.* Indeed, denote the given proper partial coloring by  $\psi$  and let  $e = uv$ . Let  $L_1$  (resp.  $L_2$ ) be the set of colors appearing on the edges incident to  $u$  (resp.  $v$ ). The coloring becomes not proper if  $e$  is colored using a color from  $L_1 \cup L_2$ , so there are  $|L_1 \cup L_2|$  such options. Suppose that coloring  $e$  with color  $c$  creates a bichromatic 4-cycle  $uvxy$ . Then  $c = \psi(xy)$  and  $\psi(vx) = \psi(uy)$ . Hence, the number of such colors  $c$  is at most the number of pairs of edges  $vx, uy$  such that  $\psi(vx) = \psi(uy)$ . Note that, since  $\psi$  is proper, there can be at most one pair  $vx, uy$  such that  $\psi(vx) = \psi(uy) = c'$  for a particular color  $c'$ . Therefore, the total number of such pairs is exactly  $|L_1 \cap L_2|$ . Thus, there are at most  $|L_1 \cup L_2| + |L_1 \cap L_2| = |L_1| + |L_2| \leq 2(\Delta - 1)$  “forbidden” colors for  $e$ , as desired.  $\dashv$

Using Claim 17, we obtain

$$\Pr(f \in F | S \in A) \leq \frac{2(\Delta - 1)}{4(\Delta - 1)} = \frac{1}{2}.$$

Therefore, if  $\omega \in \mathbb{R}_{\geq 1}$  is a constant, we get

$$\rho_\omega(f) \leq \rho_\omega(f, S) \leq \frac{\omega}{2}.$$

Now we need to deal with the edges of the form  $f_C \in E(S \cup \{e\}, S)$ . Note that there are at most  $(\Delta - 1)^{2t-2}$  cycles of length  $2t$  passing through  $e$ . Therefore, the number of edges in  $E(S \cup \{e\}, S)$  corresponding to cycles of length  $2t$  is at most  $(\Delta - 1)^{2t-2}$ . Consider any such edge  $f_C$ . Suppose that  $C = e_1, e_2, \dots, e_{2t}$ , where  $e_1 = e$ . Then  $f_C \in F$  if and only if  $\varphi(e_{2i-1}) = \varphi(e_{2t-1})$  and  $\varphi(e_{2i}) = \varphi(e_{2t})$  for all  $1 \leq i \leq t-1$ . Even if the colors of  $e_{2t-1}$  and  $e_{2t}$  are fixed, the probability of this happening is  $1/(4(\Delta - 1))^{2t-2}$ . Keeping this observation in mind, let  $C' := \{e_1, e_2, \dots, e_{2t-2}\}$ . Then

$$\Pr(f_C \in F | S \setminus C' \in A) \leq \frac{1}{(4(\Delta - 1))^{2t-2}}.$$

Also note that

$$|(S \cup \{e\}) \setminus (S \setminus C')| \leq |C'| = 2t - 2.$$

Therefore, for constant  $\omega$ , we obtain

$$\rho_\omega(f_C) \leq \rho_\omega(f_C, S \setminus C') \leq \frac{\omega^{2t-2}}{(4(\Delta - 1))^{2t-2}}.$$

Putting everything together, it is enough to find a constant  $\omega \in \mathbb{R}_{\geq 1}$  such that

$$\omega \geq 1 + \sum_{t=3}^{\infty} (\Delta - 1)^{2t-2} \cdot \frac{\omega^{2t-2}}{(4(\Delta - 1))^{2t-2}} + \frac{\omega}{2} = 1 + \frac{(\omega/4)^4}{1 - (\omega/4)^2} + \frac{\omega}{2},$$

where the last equality is valid if  $\omega/4 < 1$ . Setting  $\omega = 2(\sqrt{5} - 1)$  completes the proof.  $\blacksquare$

### 4.3 Color-critical hypergraphs

A hypergraph  $\mathcal{H}$  is  $(k+1)$ -critical if it is not  $k$ -colorable, but each of its proper subhypergraphs is. Call a hypergraph  $\mathcal{H}$  *true* if all its edges have size at least 3. It is interesting to know what the least possible number of edges in a  $(k+1)$ -critical true hypergraph on  $n$  vertices is. The best known constructions due to Abbott and Hare [1] and Abbott, Hare, and Zhou [2] contain roughly  $(k-1)n$  edges. This bound is asymptotically tight for  $k \rightarrow \infty$ , as the following theorem due to Kostochka and Stiebitz asserts.

**Theorem 18** (Kostochka, Stiebitz [23]). *Every  $(k+1)$ -critical true hypergraph with  $n$  vertices contains at least  $(k - 3k^{2/3})n$  edges.*

Here we improve this result, obtaining the following new bound.

**Theorem 19.** *Every  $(k+1)$ -critical true hypergraph with  $n$  vertices contains at least  $(k - 4\sqrt{k})n$  edges.*

*Proof.* Our proof is essentially the same as the proof of Theorem 18 given in [23]. The only difference is that we replace the application of the LLL by an application of the LCL.

Let  $\mathcal{H}$  be a  $(k+1)$ -critical true hypergraph with  $n$  vertices. Let  $c := 4\sqrt{k}$ . Fix some positive constant  $z$  (to be determined later). Let  $g : \mathbb{Z}_{\geq 1} \rightarrow \mathbb{R}$  be given by

$$g(t) := \begin{cases} 1 - z^{-1} & \text{if } t = 1; \\ 2^{1-t} z^{-1} & \text{if } t > 1. \end{cases}$$

Inductively construct a sequence  $\{V_i\}_{i=0}^m$ , where  $0 \leq m \leq n$ , of subsets of  $V(\mathcal{H})$  according to the following rule. Let  $V_0 := V(\mathcal{H})$ . If there is a vertex  $v \in V_i$  such that

$$\sum_{\substack{H \in E(\mathcal{H}): \\ v \in H}} g(|H \cap V_i|) \geq k - c, \tag{23}$$

then select one such vertex, denote it by  $v_i$ , and let  $V_{i+1} := V_i \setminus \{v_i\}$ . Otherwise let  $m := i$  and stop.

If  $m = n$ , then

$$|E(\mathcal{H})| = \sum_{H \in E(\mathcal{H})} 1 > \sum_{H \in E(\mathcal{H})} \sum_{j=1}^{|H|} g(j) = \sum_{i=0}^{n-1} \sum_{\substack{H \in E(\mathcal{H}): \\ v_i \in H}} g(|H \cap V_i|) \geq (k - c)n,$$

as desired.

Now suppose that  $m < n$ . We will prove that this cannot happen. Let  $V' := V_m$ . Since  $V'$  is nonempty, the hypergraph  $\mathcal{H} - V'$  obtained from  $\mathcal{H}$  by deleting the vertices in  $V'$  is  $k$ -colorable. Fix a proper  $k$ -coloring  $\psi$  of  $\mathcal{H} - V'$  and extend it to a  $k$ -coloring  $\varphi$  of  $\mathcal{H}$  by choosing a color for each vertex in  $V'$  uniformly and independently from all other vertices.

Consider the hypercube digraph  $Q := Q(V')$ . For  $S \subseteq V'$ , let

$$S \in A :\iff \text{there is no } \varphi\text{-monochromatic edge } H \subseteq S \cup (V \setminus V').$$

Note that  $A$  is out-closed,  $\Pr(\emptyset \in A) = 1$  (because the coloring  $\psi$  of  $V \setminus V'$  is proper), and  $V' \in A$  if and only if  $\varphi$  is a proper  $k$ -coloring of  $\mathcal{H}$ . We will use the LCL to prove that  $\Pr(V' \in A) > 0$ , which will be a contradiction since  $\mathcal{H}$  is not  $k$ -colorable.

Let  $\tilde{Q}$  be the digraph such that  $\tilde{Q}^s = Q$  and for each  $(S \cup \{v\}, S) \in E(Q)$  and  $H \in E(\mathcal{H})$  such that  $v \in H$ ,  $\tilde{Q}$  contains a corresponding edge  $e_H$  going from  $S \cup \{v\}$  to  $S$ . Let

$$e_H \in F : \iff H \text{ is } \varphi\text{-monochromatic.}$$

Then  $F$  is an  $A$ -cut.

Denote the edge set of  $\tilde{Q}$  by  $E$ . Consider some  $e_H \in E(S \cup \{v\}, S)$ . There are two cases. First suppose that  $H \not\subseteq V'$ . Note that such  $H$  is  $\varphi$ -monochromatic if and only if  $H \setminus V'$  is  $\psi$ -monochromatic and  $\varphi(u) = \psi(w)$  for all  $u \in H \cap V'$  and  $w \in H \setminus V'$ . Therefore, for each such  $H$ ,

$$\Pr(e_H \in F | S \setminus H \in A) \leq \frac{1}{k^{|H \cap V'|}}.$$

Also note that

$$|(S \cup \{v\}) \setminus (S \setminus H)| \leq |H \cap V'|.$$

Thus, if  $\omega \in \mathbb{R}_{\geq 1}$  is a constant,

$$\rho_\omega(e_H) \leq \rho_\omega(e_H, S \setminus H) \leq \frac{\omega^{|H \cap V'|}}{k^{|H \cap V'|}}.$$

If, on the other hand,  $H \subseteq V'$ , then choose an arbitrary vertex  $u \in H \setminus \{v\}$  and define  $H' := H \setminus \{u\}$ . (This definition is analogous to the one we discussed in Subsection 2.2.) We have

$$\Pr(e_H \in F | S \setminus H' \in A) \leq \frac{1}{k^{|H|-1}}$$

and

$$|(S \cup \{v\}) \setminus Z| \leq |H'| = |H| - 1,$$

so

$$\rho_\omega(e_H) \leq \rho_\omega(e_H, S \setminus H') \leq \frac{\omega^{|H|-1}}{k^{|H|-1}}.$$

For a vertex  $v \in V'$ , let

$$a_t(v) := |\{H \in E(\mathcal{H}) : v \in H \not\subseteq V', |H \cap V'| = t\}|;$$

$$b_t(v) := |\{H \in E(\mathcal{H}) : v \in H \subseteq V', |H| = t\}|.$$

To apply the LCL, it is enough to guarantee that there exists a constant  $\omega \in \mathbb{R}_{\geq 1}$  such that for all  $v \in V'$ ,

$$\omega \geq 1 + \sum_{t=1}^{\infty} a_t(v) \frac{\omega^t}{k^t} + \sum_{t=3}^{\infty} b_t(v) \frac{\omega^{t-1}}{k^{t-1}}. \quad (24)$$

Note that, since  $V'$  is the last set in the sequence  $\{V_i\}_{i=0}^m$ , no vertex in  $V'$  satisfies (23). In other words, for all  $v \in V'$ ,

$$\sum_{t=1}^{\infty} a_t(v) g(t) + \sum_{t=3}^{\infty} b_t(v) g(t) < k - c. \quad (25)$$

Let

$$\alpha_t(v) := a_t(v) g(t);$$

$$\beta_t(v) := b_t(v) g(t).$$

Then (25) can be rewritten as

$$\gamma(v) := \sum_{t=1}^{\infty} \alpha_t(v) + \sum_{t=3}^{\infty} \beta_t(v) < k - c,$$

and (24) turns into

$$\omega \geq 1 + \sum_{t=1}^{\infty} \alpha_t(v) \cdot \frac{1}{g(t)} \left(\frac{\omega}{k}\right)^t + \sum_{t=3}^{\infty} \beta_t(v) \cdot \frac{1}{g(t)} \left(\frac{\omega}{k}\right)^{t-1},$$

which, after substituting the actual values for  $g$ , becomes

$$\omega \geq 1 + \alpha_1(v) \cdot \frac{z}{z-1} \frac{\omega}{k} + \sum_{t=2}^{\infty} \alpha_t(v) \cdot \frac{1}{2} z \left( \frac{2\omega}{k} \right)^t + \sum_{t=3}^{\infty} \beta_t(v) \cdot z \left( \frac{2\omega}{k} \right)^{t-1}. \quad (26)$$

We can view the right-hand side of (26) as a linear combination of variables  $\alpha_t(v)$ ,  $\beta_t(v)$ . If we assume that

$$\frac{4\omega}{k} \geq \frac{1}{z-1},$$

then the largest coefficient in this linear combination is  $z(2\omega/k)^2$  (the coefficient of  $\beta_3(v)$ ). Thus, it is enough to find  $\omega$ ,  $z$  satisfying the following two inequalities:

$$\frac{4\omega}{k} \geq \frac{1}{z-1}; \quad (27)$$

$$\omega \geq 1 + \frac{4z\omega^2(k-c)}{k^2}. \quad (28)$$

(Inequality (28) is obtained by replacing all coefficients on the right hand side of (26) by the largest one and using the fact that  $\gamma(v) < k - c$ .) If we choose

$$z = \frac{k}{4\omega} + 1,$$

then (27) is satisfied, while (28) becomes

$$\omega \geq 1 + \frac{4\omega^2(k-c)}{k^2} \left( \frac{k}{4\omega} + 1 \right) = 1 + \frac{k-c}{k} \omega + \frac{4(k-c)}{k^2} \omega^2.$$

Thus, we just have to make sure that the following inequality has a solution  $\omega$ :

$$\frac{4(k-c)}{k^2} \omega^2 - \frac{c}{k} \omega + 1 \leq 0.$$

This is true if and only if  $c^2 \geq 16(k-c)$ ; in particular,  $c = 4\sqrt{k}$  works. Therefore,  $\varphi$  is a proper  $k$ -coloring of  $\mathcal{H}$  with positive probability. This contradiction completes the proof.  $\blacksquare$

#### 4.4 Choice functions

Our last example is a probabilistic corollary of the LCL. Let  $U_1, \dots, U_n$  be a collection of pairwise disjoint nonempty finite sets. A *choice function*  $F$  is a subset of  $\bigcup_{i=1}^n U_i$  such that for all  $1 \leq i \leq n$ ,  $|F \cap U_i| = 1$ . A *partial choice function*  $P$  is a subset of  $\bigcup_{i=1}^n U_i$  such that for all  $1 \leq i \leq n$ ,  $|P \cap U_i| \leq 1$ . For a partial choice function  $P$ , let

$$\text{dom}(P) := \{i : P \cap U_i \neq \emptyset\}.$$

Thus, a choice function  $F$  is a partial choice function with  $\text{dom}(F) = \{1, \dots, n\}$ .

Let  $F$  be a choice function and let  $P$  be a partial choice function. We say that  $P$  *occurs* in  $F$  if  $P \subseteq F$ , and we say that  $F$  *avoids*  $P$  if  $P$  does not occur in  $F$ . Many natural combinatorial problems can be stated using the language of choice functions. For instance, consider a graph  $G$  with vertex set  $\{1, \dots, n\}$ . Fix a positive integer  $k$  and let  $U_i := \{(i, c) : 1 \leq c \leq k\}$  for each  $1 \leq i \leq n$ . For each edge  $ij \in E(G)$  and  $1 \leq c \leq k$ , define a partial choice function  $P_{ij}^c := \{(i, c), (j, c)\}$ . Then a proper vertex  $k$ -coloring of  $G$  can be identified with a choice function  $F$  such that none of  $\{P_{ij}^c\}_{ij \in E(G), 1 \leq c \leq k}$  occur in  $F$ .

A *multichoice function*  $M$  is simply a subset of  $\bigcup_{i=1}^n U_i$  (one should think of it as a generalized choice function where one is allowed to choose multiple or zero elements from each set). Again, we say that a partial choice function  $P$  *occurs* in a multichoice function  $M$  if  $P \subseteq M$ . Suppose that we are given a family  $P_1, \dots, P_m$  of nonempty “forbidden” partial choice functions. For a multichoice function  $M$ , the  $i^{\text{th}}$  *defect* of  $M$  (notation:  $\text{def}_i(M)$ ) is the number of indices  $j$  such that  $i \in \text{dom}(P_j)$  and  $P_j$  occurs in  $M$ . Observe that there exists a choice function  $F$  that avoids all of  $P_1, \dots, P_m$  if and only if there exists a multichoice function  $M$  such that for all  $1 \leq i \leq n$ ,

$$|M \cap U_i| \geq 1 + \text{def}_i(M). \quad (29)$$

Indeed, if  $F$  avoids all of  $P_1, \dots, P_m$ , then  $F$  itself satisfies (29). On the other hand, if  $M$  satisfies (29), then, for every  $i$ , there is an element  $x_i \in M \cap U_i$  that does not belong to any  $P_j$  occurring in  $M$ . Therefore,  $\{x_i\}_{i=1}^n$  is a choice function that avoids all of  $P_1, \dots, P_m$ , as desired.

The main result of this subsection is that, in fact, it is enough to establish (29) *on average* for some random multichoice function  $M$ .

**Theorem 20.** *Let  $U_1, \dots, U_n$  be a collection of pairwise disjoint nonempty finite sets and let  $P_1, \dots, P_m$  be a family of nonempty partial choice functions. Let  $M_i \subseteq U_i$  be a random subset of  $U_i$  for each  $1 \leq i \leq n$ . Suppose that the variables  $M_i$  are mutually independent and let  $M := \bigcup_{i=1}^n M_i$ . If*

$$\mathbb{E}|M_i| \geq 1 + \mathbb{E} \text{def}_i(M) \quad (30)$$

*for all  $1 \leq i \leq n$ , then there exists a choice function  $F$  that avoids all of  $P_1, \dots, P_m$ .*

*Proof.* For  $x \in \bigcup_{i=1}^n U_i$ , let  $p(x) := \Pr(x \in M)$ . Then

$$\mathbb{E}|M_i| = \sum_{x \in U_i} p(x).$$

Since the variables  $\{M_i\}_{i=1}^n$  are independent,

$$\Pr(P_j \subseteq M) = \prod_{x \in P_j} p(x).$$

Therefore, if  $N_i := \{j : i \in \text{dom}(P_j)\}$ ,

$$\mathbb{E} \text{def}_i(M) = \sum_{j \in N_i} \Pr(P_j \subseteq M) = \sum_{j \in N_i} \prod_{x \in P_j} p(x).$$

Thus, (30) is equivalent to

$$\sum_{x \in U_i} p(x) \geq 1 + \sum_{j \in N_i} \prod_{x \in P_j} p(x). \quad (31)$$

Let  $\omega_i := \sum_{x \in U_i} p(x)$  and let  $q(x) := p(x)/\omega_i$  for all  $x \in U_i$ . Then (31) can be rewritten as

$$\omega_i \geq 1 + \sum_{j \in N_i} \left( \prod_{x \in P_j} q(x) \right) \left( \prod_{i' \in \text{dom}(P_j)} \omega_{i'} \right). \quad (32)$$

Construct a random choice function  $F$  as follows: Choose an element  $x \in U_i$  with probability  $q(x)$ , making the choices for different  $U_i$ 's independently (notice that  $\sum_{x \in U_i} q(x) = 1$ , so this definition is correct). Let  $Q$  be the hypercube digraph  $Q(\{1, \dots, n\})$ . For  $S \subseteq \{1, \dots, n\}$ , let

$$S \in A : \Longleftrightarrow \text{no } P_j \text{ with } \text{dom}(P_j) \subseteq S \text{ occurs in } F.$$

Then  $A$  is out-closed in  $Q$ ,  $\Pr(\emptyset \in A) = 1$ , and if  $\{1, \dots, n\} \in A$ , then  $F$  avoids all of  $P_1, \dots, P_m$ .

Let  $\tilde{Q}$  be the digraph with  $\tilde{Q}^s = Q$  and such that for each  $(S \cup \{i\}, S) \in E(Q)$  and  $j \in N_i$ ,  $\tilde{Q}$  contains a corresponding edge  $e_j$  between  $S \cup \{i\}$  and  $S$ . Define

$$e_j \in \Phi : \Longleftrightarrow P_j \text{ occurs in } F.$$

Then  $\Phi$  is an  $A$ -cut in  $\tilde{Q}$ . Since the choices for different  $U_i$ 's are independent,

$$\Pr(e_j \in \Phi | S \setminus \text{dom}(P_j) \in A) = \Pr(P_j \subseteq F | S \setminus \text{dom}(P_j) \in A) \leq \Pr(P_j \subseteq F) = \prod_{x \in P_j} q(x).$$

Let  $\omega(S \cup \{i\}, S) := \omega_i$ . Then

$$\underline{\omega}(e_j, S \setminus \text{dom}(P_j)) \leq \prod_{i' \in \text{dom}(P_j)} \omega_{i'},$$

so

$$\rho_\omega(e_j) \leq \rho_\omega(e_j, S \setminus \text{dom}(P_j)) \leq \left( \prod_{x \in P_j} q(x) \right) \left( \prod_{i' \in \text{dom}(P_j)} \omega_{i'} \right).$$

Therefore, (32) implies (2) for this  $\omega$ , meaning that, due to Corollary 6,  $\Pr(\{1, \dots, n\} \in A)$  is positive, as desired.  $\blacksquare$



Note that Theorem 20 can be used, for instance, to derive condition (11) for 2-colorability of uniform hypergraphs.

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